Coverability in VASS Revisited: Improving Rackoff's Bound to Obtain Conditional Optimality

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Problem Statement

Vector Addition Systems with States can be seen as a directed graphs with integer vector labels equipped with d non-negative counters. A configuration is a state and d natural numbers. In runs, the integer vectors are added to the counters. The size of VASS in unary encoding is denoted by n.

Coverability problem
Input: A VASS $\mathcal{V}$, an initial configuration $p(\mathbf{u})$, a target configuration $q(\mathbf{v})$.
Question: Does there exist a run in $\mathcal{V}$ from $p(\mathbf{u})$ to $q(\mathbf{w})$ where $\mathbf{w} \geq \mathbf{v}$?

Contributions

[Best paper award for Track B at ICALP'23]

❶ Coverability in VASS is always witnessed by $n^{2^{\mathcal{O}(d)}}$ length runs.
$\Longrightarrow$ Coverability in VASS can be decided in $2^{\mathcal{O}(d)} \cdot \log(n)$-space. **Optimal!**
$\Longrightarrow$ Coverability in VASS can be decided in $n^{2^{\mathcal{O}(d)}}$-time. **Conditionally Optimal!**

❷ Assuming ETH, coverability in VASS requires $n^{2^{\Omega(d)}}$-time.

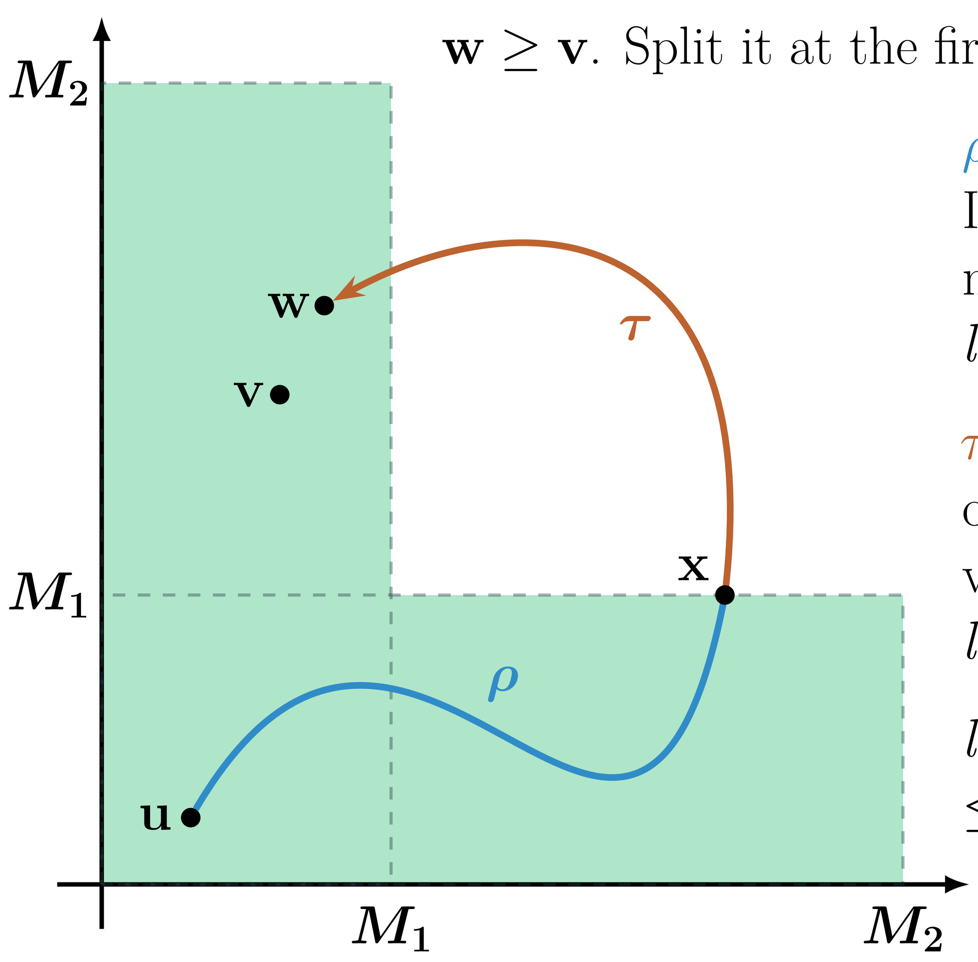
❸ Under the k-cycle hypothesis, coverability in (unary) 2-VASS requires $n^{2-o(1)}$-time.

❹ Under the k-hyperclique hypothesis, coverability in linearly-bounded VASS requires $n^{d-2-o(1)}$-time.

❶ Improving Rackoff's Upper Bound

Theorem: Coverability in VASS is always witnessed by $n^{2^{\mathcal{O}(d)}}$ length runs.
Idea: Carefully use Rackoff's technique with sharper bounds. Induction on the dimension d, if a counter exceeds a large value it can be 'ignored'.

Consider the shortest coverability run from $\mathbf{u}$ to $\mathbf{w}$ where $\mathbf{w} \geq \mathbf{v}$. Split it at the first *non-thin* configuration $\mathbf{x}$.



ρ is the *thin part* of the run. Its length is bounded by the number of thin configurations. $len(\rho) \leq d!(M_1 \cdot \ldots \cdot M_d)$

τ is the *tail* of the run. At least one component had a large value at $\mathbf{x}$, so can be 'ignored'. $len(\tau) \leq M_d$

$len(\pi) = len(\rho) + len(\tau) \leq 2 \cdot d!(M_1 \cdot \ldots \cdot M_d) \leq n^{2^{\mathcal{O}(d)}}$

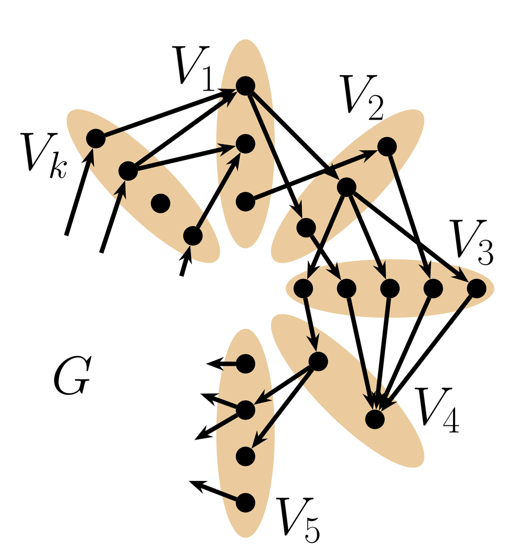
Lemma: A d-VASS can be simulated by $(d+3)$-VAS (without states).
[Hopcroft and Pansiot '79]

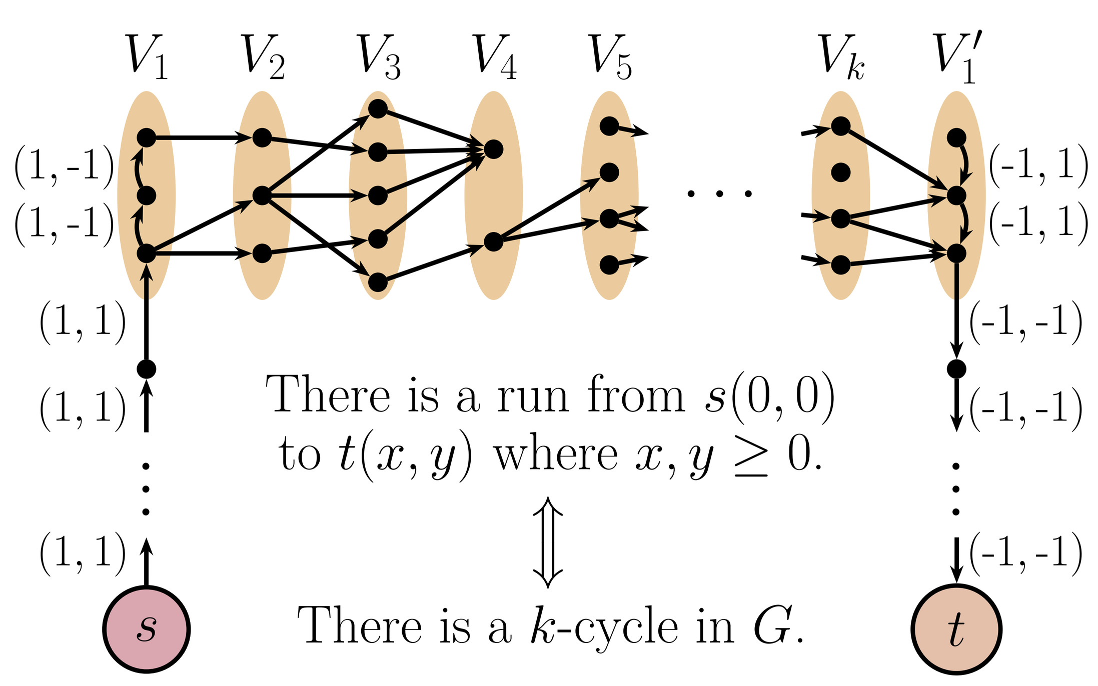
❸ Coverability in Unary 2-VASS

Hypothesis: Finding a k-cycle in a k-circle-layered graph of m edge requires $m^{2-o(1)}$-time. [Lincoln, Vassilevska Williams, Williams '18]

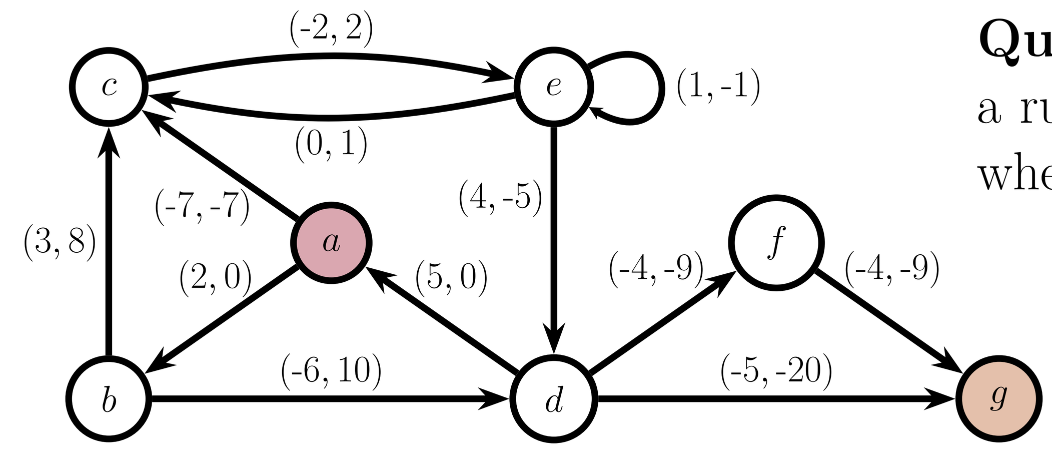
Theorem: Under the k-cycle hypothesis, coverability in (unary) 2-VASS requires $n^{2-o(1)}$-time.

Reduction sketch:





Example Instance



Question: Does there exist a run from $a(0,0)$ to $g(x,y)$ where $x, y \geq 0$?

Brief History & Complexity

Theorem: Coverability in VASS requires $2^{\Omega(d)} \cdot \log(n)$-space.
Idea: Find instance only admitting $n^{2^{\Omega(d)}}$ length runs. *"Lipton's construction"*
[Lipton '76] [Rosier and Yen '85]

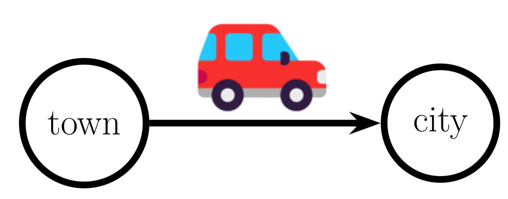
Theorem: Coverability in VASS can be decided in $2^{\mathcal{O}(d \log d)} \cdot \log(n)$-space.
Idea: Argue that there are always $n^{2^{\mathcal{O}(d \log d)}}$ length runs. *"Rackoff's bound"*
[Rackoff '78] [Rosier and Yen '85]

Open problem: Improve these bounds. [Mayr and Meyer '82]

Motivation

Resource management:

Finding routes for multi-fuel vechicles.

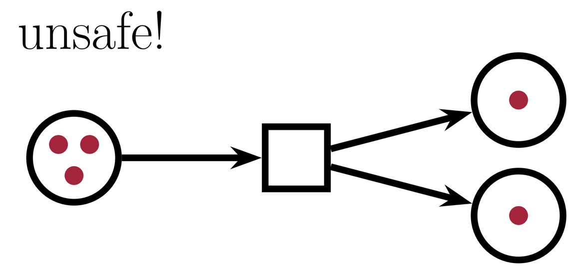


(−1L fuel, +2kWh battery)

Testing safety: A positive instance of coverability.
$\Longrightarrow$ Some sequence of actions reaches a 'bad' state.
$\Longrightarrow$ The given system is unsafe!

Model of concurrency:

VASS are equivalent to Petri nets.



Related problems: Boundedness, reachability, and word problems for (commutative) semi-groups.

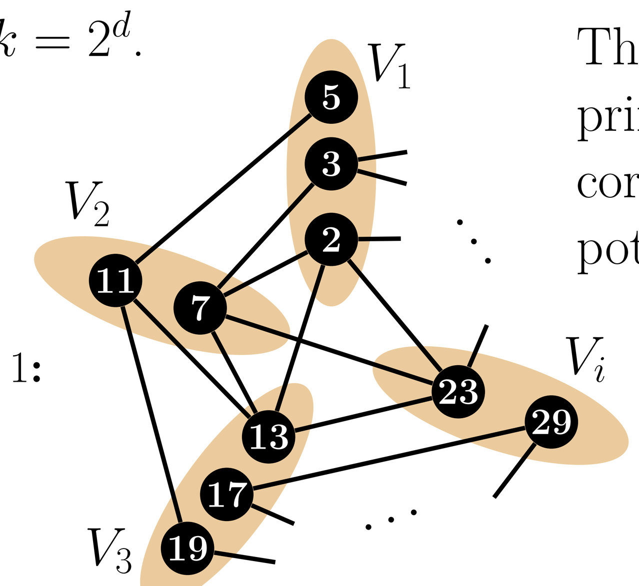
❷ Conditional Time Lower Bound

Exponential **T**ime **H**ypothesis: 3-SAT requires $2^{\Omega(n)}$-time.
$\Longrightarrow$ Finding a k-clique in a k-partite graph of n verticies requires $n^{\Omega(k)}$-time.
Theorem: Assuming ETH, coverability in VASS requires $n^{2^{\Omega(d)}}$-time.

Reduction sketch: Let $k = 2^d$.

Counter program

For $i = 1$ **to** k:
  Guess: $v \in \{1, \ldots, n\}$
  $\mathtt{x} \leftarrow \mathtt{x} \cdot p_v$
For $i = 1$ **to** k **and** $j = 1$ **to** $i - 1$:
  Guess: $\{u, v\} \in E \cap (V_i \times V_j)$
  $\mathtt{x} \leftarrow \mathtt{x} \div (p_u \cdot p_v)$
  $\mathtt{x} \leftarrow \mathtt{x} \cdot (p_u \cdot p_v)$



The product of k primes uniquely corresponds to a potential clique.

Can be implemented as a $\mathcal{O}(n^k)$-bounded 2-VASS with zero-tests.

That can be simulated by a $\mathcal{O}(\log(k))$-VASS. [Rosier and Yen '85]

❹ Coverability in Linearly-Bounded VASS

Hypothesis: Finding a k-hyperclique in a 3-uniform k-partite hypergraph of n verticies requires $n^{k-o(1)}$-time. [Lincoln, Vassilevska Williams, Williams '18]

Linearly-bounded VASS have their maximum counter values bounded above by a constant multiple of the size of the VASS.

Observation: Coverability in linearly-bounded VASS can be decided in $\mathcal{O}(n^{d+1})$-time by a trivial exhaustive search of the configurations.

Theorem: Under the k-hyperclique hypothesis, coverability in linearly-bounded VASS requires $n^{d-2-o(1)}$-time.